

RICCATI TECHNIQUES AND VARIATIONAL PRINCIPLES IN OSCILLATION THEORY FOR LINEAR SYSTEMS

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ABSTRACT. We consider the second order differential system (1) $Y'' + Q(t)Y = 0$, where Q, Y are $n \times n$ matrices with $Q = Q(t)$ a continuous symmetric matrix-valued function, $t \in [a, +\infty)$. We obtain a number of sufficient conditions in order that all prepared solutions $Y(t)$ of (1) are oscillatory. Two approaches are considered, one based on Riccati techniques and the other on variational techniques, and involve assumptions on the behavior of the eigenvalues of $Q(t)$ (or of its integral). These results extend some well-known averaging techniques for scalar equations to system (1).

1. Introduction. Consider the second order differential system

$$(1.1) \quad Y'' + Q(t)Y = 0, \quad t \in [a, +\infty),$$

where $Y(t)$, $Q(t)$ are $n \times n$ real continuous matrix functions with $Q(t)$ symmetric. The vector system associated with (1.1) is

$$(1.2) \quad y'' + Q(t)y = 0$$

where $y = \text{col}(y_1, \dots, y_n)$ is an n -vector. Equation (1.2) is said to be *disconjugate* on an interval $J \subset [a, +\infty)$ if every nontrivial solution of (1.2) vanishes at most once in J and (1.2) is said to be *oscillatory* if for each $t_0 > a$ there exists $t_1 > t_0$ such that (1.2) is not disconjugate on $[t_0, t_1]$. A solution $Y(t)$ of the matrix equation (1.1) is said to be *nontrivial* if $\det Y(t) \neq 0$ for at least one $t \in [a, +\infty)$ and a nontrivial solution $Y(t)$ is said to be *prepared* or *self-conjugate* in case

$$(1.3) \quad Y^*(t)Y'(t) - Y'^*(t)Y(t) \equiv 0, \quad t \in [a, +\infty),$$

(where for any matrix A , the transpose of A is denoted by A^*). Note that for *any* solution $Y(t)$ of (1.1) the expression on the left of (1.3) is constant. Equation (1.1) is said to be *oscillatory* on $[a, +\infty)$ in case the determinant of every nontrivial prepared solution vanishes on $[b, +\infty)$ for each $b > a$. This is equivalent to oscillation of equation (1.2) since any solution of (1.2) is of the form $y(t) \equiv Y(t)\alpha$ for some constant vector α and some nontrivial prepared solution $Y(t)$ of (1.1).

The oscillation theory for the corresponding scalar equation

$$(1.4) \quad y'' + q(t)y = 0, \quad t \in [a, +\infty),$$

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where $q(t)$ is a real-valued continuous function has a very extensive literature and there are a number of survey papers which have been written, [27, 30, 23]—see also the book of Swanson [24]. The corresponding theory for (1.1) and (1.2) is less developed although there have been quite a number of papers devoted to attempts at paralleling the scalar theory. Many recent results [13, 7, 8, 26, 11, 2, 25], have concentrated on showing that (1.1) is oscillatory if a corresponding scalar equation obtained by applying a positive linear functional is oscillatory. That is, if we denote by \mathcal{S} the linear space of all $n \times n$ real symmetric matrices, then a linear functional $\varphi: \mathcal{S} \rightarrow (-\infty, +\infty)$ is said to be *positive* if $\varphi(A) \geq 0$ for $A \in \mathcal{S}$ and $A \geq 0$ (i.e., A symmetric and positive semidefinite). The basic result obtained (cf. [11, 13]) is that (1.1) (or (1.2)) is oscillatory on $[a, +\infty)$ in case there exists a positive linear functional φ with $\varphi(I) = 1$ (I = identity matrix) such that the scalar equation

$$(1.5) \quad u'' + \varphi(Q(t))u = 0$$

is oscillatory.

Several other recent oscillation criteria for (1.1) have been given which involve the eigenvalues of $Q(t)$ (or of its integral). For any real symmetric matrix A , we will assume its eigenvalues $\lambda_k(A)$, $1 \leq k \leq n$, are ordered so that

$$(1.6) \quad \lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A),$$

and as usual,

$$(1.7) \quad \operatorname{tr} A = \sum_{i=1}^n \lambda_i(A).$$

For the scalar equation (1.4) the well-known Fite-Wintner Theorem [9, 30] states that (1.4) is oscillatory if

$$(1.8) \quad \int_a^\infty q(t) dt = +\infty$$

and hence it is clear that (1.1) is oscillatory if

$$(1.9) \quad \int_a^\infty \operatorname{tr} Q(t) dt = +\infty.$$

Mingarelli in [18, 19] showed that the positive *linear* functional $\operatorname{tr} A$ can be replaced by the *convex* function $\lambda_1(A)$, provided a certain growth condition on $\int_a^t \operatorname{tr} Q(s) ds$ is assumed. This answered, in part, a conjecture [13] which was also considered in the thesis of Akiyama [1]. Additional results which extend and improve some of these criteria were also obtained in [4, 5, 14, 17].

Finally, it has recently been shown by Byers, Harris and Kwong [6] that

$$(1.10) \quad \lim_{t \rightarrow \infty} \lambda_1 \left(\int_a^t Q(s) ds \right) = +\infty$$

without any additional conditions, is an oscillation criterion for (1.1). This gives, therefore, the desired systems analogue of the Fite-Wintner condition.

It is the aim of this paper to apply two of the fundamental approaches used in studying the scalar equation (1.4)—namely the Riccati integral equation approach and the variational approach—to obtain a number of oscillation tests for (1.1). We

refer to the recent work of Kwong and Zettl [15] and the references therein where these two approaches are considered for the scalar equation.

In §§2 and 3 we present several oscillation criteria which may be regarded as generalizations to systems of various well-known scalar tests, other than the Fite-Wintner criterion. In §2 the proofs are based on the Riccati equation technique, whereas in §3 we discuss oscillation criteria which are proved using variational techniques. In §4 we present several examples showing that certain subsidiary hypotheses for these theorems are necessary. The proofs of the results are given in §§5 and 6.

In a strictly formal sense, any oscillation or nonoscillation theorem for (1.1) should be accessible by either the Riccati or variational method. In practice, by considering these two different approaches, one seems to gain greater insight into the way in which conditions imposed on the coefficient matrix Q or its integral contribute to the oscillatory nature of the equation. This is one of the purposes of this present paper. Our other purpose is to illustrate the extent to which scalar oscillation criteria can be generalized to systems, utilizing the behavior of the largest eigenvalue and thereby achieving sharper results than can be obtained by the use only of positive linear functionals such as the trace.

We recall here some notation and definitions which will be useful in the sequel. For any subset E of the real line R , $\mu(E)$ denotes the Lebesgue measure of E . If $f(t)$ denotes a continuous real-valued function and if l, m satisfy $-\infty \leq l, m \leq +\infty$, then we say $\lim \text{approx} \inf_{t \rightarrow \infty} f(t) = l$ in case $\mu\{t: f(t) \leq l_1\} < +\infty$ for all $l_1 < l$ and $\mu\{t: f(t) \leq l_2\} = +\infty$ for all $l_2 > l$. Similarly, $\lim \text{approx} \sup_{t \rightarrow \infty} f(t) = m$ in case $\mu\{t: f(t) \geq m_1\} = +\infty$ for all $m_1 < m$ and $\mu\{t: f(t) \geq m_2\} < +\infty$ for all $m_2 > m$. Finally, $\lim \text{approx}_{t \rightarrow \infty} f(t) = \lambda$ in case

$$\lim \text{approx} \sup_{t \rightarrow \infty} f(t) = \lim \text{approx} \inf_{t \rightarrow \infty} f(t) = \lambda.$$

2. Riccati techniques. We recall [12] that the scalar equation (1.4) is nonoscillatory on $[a, +\infty)$ if and only if the Riccati integral equation

$$(2.1) \quad r(t) = r(t_0) + \int_{t_0}^t q(s) ds + \int_{t_0}^t r^2(s) ds$$

has a continuous solution on $[t_0, \infty)$ for some $t_0 \geq a$; likewise the matrix equation (1.1) is nonoscillatory on $[a, +\infty)$ if and only if the corresponding matrix equation

$$(2.2) \quad Z(t) = Z(t_0) + \int_{t_0}^t Q(s) ds + \int_{t_0}^t Z^2(s) ds$$

has a continuous symmetric solution on $[t_0, \infty)$ for some $t_0 \geq a$.

If (1.4) has a solution $y = y(t)$ with $y(t) \neq 0$ for $t \geq t_0$, then $r = -y'/y$ is a solution of (2.1) on $[t_0, \infty)$; similarly, if $Y = Y(t)$ is a nontrivial prepared solution of (1.1) with $\det Y(t) \neq 0$ for $t \geq t_0$ then $Z = -Y'Y^{-1}$ is a symmetric solution of (2.2) on $[t_0, \infty)$.

Hartman [11] has shown that if (1.4) is nonoscillatory on $[a, +\infty)$ then a necessary and sufficient condition that

$$(2.3) \quad \int_0^\infty \left(\frac{u'}{u} \right)^2 dt < \infty$$

holds for a solution $u \not\equiv 0$ of (1.4) is that

$$(2.4) \quad \liminf_{T \rightarrow \infty} \frac{1}{T} \int_a^T \int_a^t q(s) \, ds \, dt > -\infty.$$

One can therefore consider the cases where (2.4) does or does not hold when studying the oscillatory behavior of (1.4).

As an extension of the Fite-Wintner criterion (1.8), Olech, Opial, and Wazewski in [21] showed that (1.4) is oscillatory in case

$$(2.5) \quad \lim_{T \rightarrow \infty} \text{approx} \int_a^T q(s) \, ds = +\infty$$

or in case

$$(2.6) \quad \lim_{T \rightarrow \infty} \text{approx inf} \int_a^T q(s) \, ds < \lim_{T \rightarrow \infty} \text{approx sup} \int_a^T q(s) \, ds.$$

Wintner in [29] showed that

$$(2.7) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_a^T \int_a^t q(s) \, ds \, dt = +\infty$$

implies oscillation of (1.4) and Hartman [12] showed that

$$(2.8) \quad -\infty < \liminf_{T \rightarrow \infty} \frac{1}{T} \int_a^T \int_a^t q(s) \, ds \, dt < \limsup_{T \rightarrow \infty} \frac{1}{T} \int_a^T \int_a^t q(s) \, ds \, dt$$

is also sufficient for oscillation of (1.4). These results were further generalized by Willett [27, 28] via more general averaging techniques.

Analogous to (2.4), we consider the (extended real-valued) function L which is defined on the class of $n \times n$ continuous real symmetric matrices defined on $[a, \infty)$ by

$$(2.9) \quad L(Q) \equiv \liminf_{T \rightarrow \infty} \frac{1}{T} \int_a^T \int_a^t \text{tr} \, Q(s) \, ds \, dt.$$

We now state the main results of this section.

THEOREM 2.1. *Assume $L(Q) > -\infty$. Then equation (1.1) is oscillatory in case any of the following conditions hold:*

$$(A) \quad \limsup_{T \rightarrow \infty} \frac{1}{T} \int_a^T \lambda_1 \left(\int_a^t Q(s) \, ds \right) dt = +\infty,$$

$$(B) \quad \limsup_{T \rightarrow \infty} \frac{1}{T} \int_a^T \left[\lambda_1 \left(\int_a^t Q(s) \, ds \right) \right]^2 dt = +\infty,$$

$$(C) \quad \lim_{T \rightarrow \infty} \text{approx sup} \lambda_1 \left(\int_a^T Q(s) \, ds \right) = +\infty,$$

$$(D) \quad \lim_{T \rightarrow \infty} \text{approx inf} \lambda_1 \left(\int_a^T Q(s) \, ds \right) = -\infty.$$

THEOREM 2.2. Assume $L(Q) = -\infty$. Then (1.1) is oscillatory if

$$(2.10) \quad \lim_{T \rightarrow \infty} \text{approx sup } \lambda_n \left(\int_a^T Q(s) ds \right) > -\infty.$$

THEOREM 2.3. Assume that $\lambda_1(\int_a^t Q(s) ds) > 0$ for sufficiently large t , and that

$$(2.11) \quad \liminf_{t \rightarrow \infty} \left| \frac{\lambda_1(\int_a^t Q(s) ds)}{\lambda_n(\int_a^t Q(s) ds)} \right| > 0.$$

Then (1.1) is oscillatory if

$$(A^*) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_a^T \lambda_1 \left(\int_a^t Q(s) ds \right) dt = +\infty.$$

Since $\text{tr } Q$ is a positive linear functional, systems analogues for each of the above scalar criteria (1.8), (2.5)–(2.8) may be obtained if $q(t)$ is replaced by $\text{tr } Q(t)$ (see the discussion of this technique in the introduction). The intention behind Theorems 2.1 and 2.3 is to show that the corresponding (and in general, much weaker) assumptions concerning the behavior of $\lambda_1(\int_a^t Q(s) ds)$ will provide oscillation criteria for the system (1.1). We observe then, that Theorems 2.1(A) and 2.3 are generalizations of the results (2.8) of Hartman and a partial generalization of the result (2.7) of Wintner, respectively, and Theorems 2.1(C), (D) are partial generalizations of the oscillation criteria results (2.5), (2.6) of Olech et al. The scalar version of Theorem 2.1(B), namely that condition (2.4) and

$$(2.12) \quad \limsup_{T \rightarrow \infty} \frac{1}{T} \int_a^T \left(\int_a^t q(s) ds \right)^2 dt = \infty$$

imply oscillation, although a fairly simple result, does not appear to be explicitly stated in the literature. The scalar analogue of Theorem 2.2 also does not appear to be stated explicitly elsewhere, although it is related to results of Kwong and Zettl [15].

As was mentioned in §1, the systems analogue of the Fite-Wintner theorem has been recently obtained [6]. That a complete analogue of the corresponding scalar oscillation theorem employing only the behavior of λ_1 is not always available for systems will be shown for the Wintner criterion (2.7) in §4.

3. Variational techniques. In this section, we turn our attention to the use of variational principles for obtaining oscillation criteria for (1.1). We assume that $Q(t)$ is symmetric and locally integrable on $[a, \infty)$. For any subinterval $[\alpha, \beta]$ of $[a, \infty)$, define $A_1(\alpha, \beta)$ to be

$$\{\eta: [\alpha, \beta] \rightarrow \mathbf{R}^n | \eta(\alpha) = \eta(\beta) = 0, \eta \in AC[\alpha, \beta], \eta' \in L^2(\alpha, \beta)\}.$$

The basic result we need here [12] is that (1.1) is oscillatory if and only if there is a sequence of intervals $[a_n, b_n]$, with $\lim_{n \rightarrow \infty} a_n = \infty$, and a sequence of functions $\varphi_n \in A_1(a_n, b_n)$, such that

$$(3.1) \quad \int_{a_n}^{b_n} \{|\varphi'_n(t)|^2 - \varphi_n^*(t)Q(t)\varphi_n(t)\} dt < 0.$$

The idea then is to utilize conditions on the behavior of Q or its integral in order to construct sequences fulfilling (3.1).

To illustrate this technique, we present three oscillation criteria and a comparison theorem for oscillation. Theorem 3.1 is a generalization of a result of Moore [20] in the scalar case and Theorem 3.3 is a generalization of a result of Olech et al. [21] in the scalar case (see (2.8) and also Theorem 2.1(c) in §2). The comparison result, Theorem 3.4, is an extension to systems of the “telescoping” principle of Kwong and Zettl [16]. Theorem 3.2 has no scalar analogue.

THEOREM 3.1. *Suppose that*

$$(3.2) \quad \limsup_{t \rightarrow \infty} \lambda_1 \left(\int_a^t Q(s) ds \right) = \infty.$$

Then (1.1) is oscillatory if either

$$(A) \quad \lambda_1(Q(t)) \text{ is bounded above on } [a, \infty),$$

or

$$(B) \quad \lambda_n(Q(t)) \text{ if bounded below on } [a, \infty).$$

THEOREM 3.2. *Let the entries of $Q(t)$ be $q_{ij}(t)$; $i, j = 1, \dots, n$. Assume that each $q_{ij}(t)$ is bounded on $[a, \infty)$ and $q_{ii} \equiv 0$, $i = 1, \dots, n$.*

Then (1.1) is oscillatory if there exist i, j such that

$$(3.3) \quad \limsup_{t \rightarrow \infty} \int_a^t q_{ij}(s) ds = \infty.$$

THEOREM 3.3. *Suppose that for each positive integer $m \geq a$, there exists a positive number ε_m , and for each positive integer k , there exists a unit vector $x_{mk} \in \mathbf{R}^n$ such that the set*

$$S_{mk} = \left\{ t \geq m: x_{mk}^* \left(\int_m^t Q(s) ds \right) x_{mk} \geq k \right\}$$

has measure at least ε_m . Then (1.1) is oscillatory.

In order to state the next result, we need to introduce classes of “telescoping” transformations discussed for the scalar case in [16]. For any (extended) real numbers α, β with $\alpha < \beta \leq \infty$ and any natural number n , let $D_\alpha^\beta(n)$ denote the set of piecewise continuous $n \times n$ matrix-valued functions on $[\alpha, \beta)$ and let $S = \bigcup_{i=1}^N (\alpha_i, \beta_i)$ be a denumerable (finite or infinite) union of subintervals of $[\alpha, \beta)$ such that

$$(3.4) \quad \alpha < \alpha_i < \beta_i < \alpha_{i+1} < \beta \quad \text{for each } i.$$

If $Q \in D_\alpha^\beta(n)$, then $T_S(Q)$ is defined to be a piecewise continuous $n \times n$ matrix-valued function \tilde{Q} , defined on some interval $[\alpha, A)$, which is obtained by “collapsing” each interval (α_i, β_i) to a point. Here,

$$\begin{aligned} A &= \sum_{i=1}^N (\alpha_i - \beta_{i-1}), \quad \text{if } \lim_{i \rightarrow N} \beta_i = \beta \ (\beta_0 = \alpha), \\ &= \sum_{i=1}^N (\alpha_i - \beta_{i-1}) + \beta - l, \quad \text{if } \lim_{i \rightarrow N} \beta_i = l < \beta. \end{aligned}$$

We refer to [16] for further details.

THEOREM 3.4. Let $S = \bigcup_{i=1}^N (\alpha_i, \beta_i)$ be a denumerable union of subintervals of $[\alpha, \infty)$ satisfying (3.4) (with $\alpha_1 = a$, $\beta_N = \infty$). Assume that

$$(3.5) \quad \int_{\alpha_i}^{\beta_i} Q(t) dt \text{ is nonnegative definite for each } i,$$

and suppose that

$$(3.6) \quad Y'' + \tilde{Q}(t)Y = 0, \quad t \in [a, A),$$

is oscillatory, where $\tilde{Q} = T_S(Q)$. Then (1.1) is oscillatory.

4. Examples and counterexamples.

EXAMPLE 1. We wish to indicate here that in Theorem 2.1, parts (A), (B), one cannot replace the *average* of $\lambda_1(\int_a^t Q(s) ds)$ or $(\lambda_1(\int_a^t Q(s) ds))^2$ by these expressions alone, and that one cannot replace “lim approx sup” by “lim sup” in parts (C), (D). It suffices to consider a scalar counterexample. We choose $v \in C^1[a, \infty) \cap L^2[a, \infty)$ and define $q(t) = v' - v^2$. Then $y = \exp(-\int_a^t v(s) ds)$ is a nonoscillatory solution of $y'' + qy = 0$ and we have

$$(4.1) \quad \int_a^t q(s) ds = v(t) - v(a) - \int_a^t v^2(s) ds.$$

It is clear that we can choose v such that $\limsup_{t \rightarrow \infty} v(t)$ takes on any value M with $-\infty < M \leq +\infty$, $\liminf_{t \rightarrow \infty} v(t)$ takes on any value m , $-\infty \leq m \leq M \leq +\infty$, and such that $\liminf_{T \rightarrow \infty} (1/T) \int_a^T v(s) ds > -\infty$.

EXAMPLE 2. The condition

$$(4.2) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_a^t \lambda_1 \left(\int_a^s Q(\sigma) d\sigma \right) ds = \infty$$

is not by itself an oscillation criteria for (1.1) when $n > 1$. To see this, consider the case where $Q(t)$ is a 2×2 diagonal matrix for all t , say $Q(t) = \text{diag}\{q_1(t), q_2(t)\}$.

Then (1.1) uncouples as two scalar equations

$$(4.3)_i \quad y_i'' + q_i(t)y_i = 0; \quad i = 1, 2.$$

If we take $a = 0$ and denote $\int_0^t q_i(s) ds$ by $p_i(t)$, we will obtain the required counterexample if we can find $q_1(t), q_2(t)$ so that (4.3)₁ and (4.3)₂ are both nonoscillatory, with

$$(4.4) \quad p_2(t) \geq 0, \quad t \geq 0,$$

$$(4.5) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (p_1(t))_+ dt = \infty, \quad \text{where } (p_1(t))_+ = \max(0, p_1(t)).$$

For then, on choosing

$$c_1(t) = \begin{cases} 1, & t \in \text{support } (p_1(t))_+, \\ 0, & \text{otherwise,} \end{cases} \quad c_2(t) = \begin{cases} 0, & t \in \text{support } (p_1(t))_+, \\ 1, & \text{otherwise,} \end{cases}$$

and defining $x(t) = (c_1(t), c_2(t))^*$, we have

$$\begin{aligned} \lambda_1 \left(\int_0^t Q(s) ds \right) &\geq x^*(t) \left(\int_0^t Q(s) ds \right) x(t) \\ &= c_1^2(t)p_1(t) + c_2^2(t)p_2(t) \geq (p_1(t))_+. \end{aligned}$$

Thus (4.2) holds, but clearly (1.1) is nonoscillatory. The choice of $q_2(t)$ is straightforward, e.g. take $q_2(t) = 1/4(1+t)^2$. To construct a suitable $q_1(t)$, we first give an inductive definition of a suitable Riccati scalar variable $z_1(t)$. Define z_1 on $[0, 1]$ by $z_1(0) = 0$, $z_1(t) = 1$, $\frac{1}{2} \leq t \leq 1$, so that z_1 is continuously differentiable and monotone nondecreasing on $[0, 1]$. Define a_1 to be $\int_0^1 z_1^2(s) ds$, b_1 to be $4a_1$ and ε_1 to be $1/8b_1$. Then $a_1 \geq \frac{1}{2}$, $b_1 \geq 2$ and $2\varepsilon_1 \leq \frac{1}{8} < \frac{1}{2}$. Let $s_m = \sum_{i=1}^m 1/i$. Extend z_1 to $[s_1, s_2]$ by taking $z_1(t) = b_1$ on $[1 + \varepsilon_1, s_2]$ and making z_1 continuously differentiable and monotone nondecreasing on $[0, s_2]$. Inductively, suppose that a_i, b_i, ε_i have been defined for $1 \leq i \leq m$, with $b_i = 4a_i$, $\varepsilon_i = 1/8b_i < 1/2(i+1)$, and z_1 has been defined as a continuously differentiable, monotone nondecreasing function on $[0, s_{m+1}]$ such that $z_1(t) \equiv b_i$ on $[s_i + \varepsilon_i, s_{i+1}]$ and $\int_0^{s_i} z_1^2(s) ds = a_i$, $1 \leq i \leq m$. Then

$$a_{m+1} \geq a_m + \int_{s_m + \varepsilon_m}^{s_m + 2\varepsilon_m} z_1^2(s) ds = a_m + b_m^2 \varepsilon_m \geq \frac{3}{2} a_m.$$

By the inductive hypothesis, we see that $a_{m+1} \geq (3/2)^m a_1 \geq \frac{1}{2}(3/2)^m$. If $b_{m+1} = 4a_{m+1}$, $\varepsilon_{m+1} = 1/8b_{m+1}$, then $\varepsilon_{m+1} \leq (3/2)^{-m}/16 \leq 1/2(m+2)$ (since $(3/2)^m = (1 + 1/2)^m \geq 1 + m/2 > 2(m+2)/16$), and we extend z_1 to $[s_{m+1}, s_{m+2}]$ so that $z_1(t) \equiv b_{m+1}$ on $[s_{m+1} + \varepsilon_{m+1}, s_{m+2}]$ and is continuously differentiable and monotone nondecreasing on $[0, s_{m+2}]$. With this inductive definition of z_1 on $[0, \infty)$, note that if $t \in [s_m + \varepsilon_m, s_m + 2\varepsilon_m]$, we have

$$(4.6) \quad z_1(t) - \int_0^t z_1^2(s) ds \geq b_m - a_m - 2\varepsilon_m b_m^2 \geq \frac{1}{2} b_m.$$

Define $q_1(t)$ to be $z_1'(t) - z_1^2(t)$. Then for $t \in [s_k + \varepsilon_k, s_k + 2\varepsilon_k]$, we have

$$(4.7) \quad p_1(t) = \int_0^t q_1(s) ds \geq \frac{1}{2} b_k.$$

Thus if $s_m \leq t \leq s_{m+1}$, we have

$$(4.8) \quad \frac{1}{t} \int_0^t (p_1(s))_+ ds \geq \frac{1}{t} \sum_{k=1}^m \frac{1}{2} b_k \cdot \varepsilon_k = \frac{m}{16t}.$$

Since $s_m \sim \log m$, for $s_m \leq t \leq s_{m+1}$ we have $m \sim e^t$ so that

$$(4.9) \quad \frac{1}{t} \int_0^t (p_1(s))_+ ds \sim \frac{e^t}{16t}.$$

Thus (4.5) holds, yet (4.3)₁ is nonoscillatory, and so the example is complete.

EXAMPLE 3. Akiyama [1] conjectured that a condition weaker than (1.10), namely

$$(4.10) \quad \lim_{t \rightarrow \infty} \int_0^t \lambda_1(Q(s)) ds = \infty$$

might suffice for oscillation of (1.1).

The previous example may be used to provide a counterexample to this conjecture. For if we define $q_1(t)$, $q_2(t)$, $z_1(t)$, etc. as in that example, then

$$(4.11) \quad \int_0^t \lambda_1(Q(s)) ds \geq \int_0^t (q_1(s))_+ ds = \int_0^t (z_1'(s) - z_1^2(s))_+ ds.$$

Now on $[s_m, s_m + \varepsilon_m]$, we have $z_1(s) \leq b_m$, and so

$$\begin{aligned} \int_{s_m}^{s_m + \varepsilon_m} (z_1'(s) - z_1^2(s))_+ ds &\geq \int_{s_m}^{s_m + \varepsilon_m} (z_1'(s) - z_1^2(s)) ds \\ &= z_1(s_m + \varepsilon_m) - z_1(s_m) - \int_{s_m}^{s_m + \varepsilon_m} z_1^2(s) ds \\ &\geq b_m - b_{m-1} - b_m^2 \varepsilon_m \geq 4(a_m - a_{m-1}) - \frac{1}{8} b_m \\ &= \frac{4}{3} a_m - \frac{1}{8} b_m = \frac{5}{24} b_m, \end{aligned}$$

and so

$$\int_0^{s_{m+1}} \lambda_1(Q(s)) ds \geq \frac{5}{24} \sum_{k=1}^m b_k \rightarrow \infty \quad \text{as } m \rightarrow \infty.$$

5. Proofs for §2. The proofs of Theorem 2.1 and Theorem 2.2 will make use of the following lemma which is the systems analogue of the result of Hartman mentioned in §2 (cf. (2.3), (2.4)).

LEMMA 5.1. *Assume equation (1.1) is nonoscillatory on $[a, +\infty)$. Then a necessary and sufficient condition that*

$$(5.1) \quad \lim_{T \rightarrow \infty} \int_t^T Z^2(s) ds$$

exists for any solution $Z = -Y'Y^{-1}$ of (2.2) where $Y(t)$ is a prepared solution of (1.1), is that

$$(5.2) \quad L(Q) = \liminf_{T \rightarrow \infty} \frac{1}{T} \int_a^T \int_a^t \text{tr } Q(s) ds dt > -\infty.$$

PROOF. Suppose first that the limit in (5.1) exists for some solution $Z = -Y'Y^{-1}$ of (2.2), where $\det Y(t) \neq 0$, $t \geq a$, so that we have from (2.2)

$$(5.3) \quad \text{tr } Z(t) + \text{tr} \int_t^\infty Z^2(s) ds = \int_a^t \text{tr } Q(s) ds - C,$$

where $C = -\text{tr } Z(a) - \text{tr} \int_a^\infty Z^2(s) ds$. Since

$$\begin{aligned} (5.4) \quad &\frac{1}{T} \int_a^T \left(\text{tr } Z(t) - \int_t^\infty \text{tr } Z^2(s) ds \right)^2 dt \\ &\leq \frac{2}{T} \left[\int_a^T \left((\text{tr } Z(t))^2 + \left(\int_t^\infty \text{tr } Z^2(s) ds \right)^2 \right) dt \right] \end{aligned}$$

and since $\text{tr } Z(t)^2 \leq n \text{tr } Z^2(t)$, we have $(1/T) \int_a^T \text{tr } Z^2(t) dt \rightarrow 0$ as $T \rightarrow \infty$. Similarly, $(1/T) \int_a^T (\int_t^\infty \text{tr } Z^2(s) ds)^2 dt \rightarrow 0$ as $T \rightarrow \infty$ so that (5.3) and (5.4) imply

$$(5.5) \quad \frac{1}{T} \int_a^T \left[\int_a^t \text{tr } Q(s) ds - C \right]^2 dt \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

Now

$$\left| \frac{1}{T} \int_a^T \left[\int_a^t \operatorname{tr} Q(s) ds - C \right] dt \right| \leq \left(\frac{1}{T} \int_a^T \left[\int_a^t \operatorname{tr} Q(s) ds - C \right]^2 dt \right)^{1/2}$$

by the Cauchy-Schwarz inequality and therefore it follows that

$$(5.6) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_a^T \left[\int_a^t \operatorname{tr} Q(s) ds \right] dt = C \quad \text{exists,}$$

so that (5.2) holds.

Conversely, suppose (5.2) holds and let $Z = -Y'Y^{-1}$, where $Y(t)$ is a prepared solution of (1.1) with $\det Y(t) \neq 0$, $t \geq a$. From (2.2) we have

$$(5.7) \quad \begin{aligned} -\frac{1}{T} \int_a^T Z(t) dt + \frac{1}{T} \int_a^T \int_a^t Z^2(s) ds dt \\ = -\frac{1}{T}(T-a)Z(a) - \frac{1}{T} \int_a^T \int_a^t Q(s) ds dt \end{aligned}$$

so that from (5.2) we have

$$(5.8) \quad -\frac{1}{T} \int_a^T \operatorname{tr} Z(s) ds + \frac{1}{T} \int_a^T \int_a^t \operatorname{tr} Z^2(s) ds dt \leq M, \quad \text{for some } M > 0.$$

Since $\operatorname{tr} Z^2(t) \geq 0$, it follows that $\lim_{t \rightarrow \infty} \int_a^t \operatorname{tr} Z^2(s) ds$ exists, finite or infinite.

Suppose that $\int_a^t \operatorname{tr} Z^2(s) ds \rightarrow +\infty$ as $t \rightarrow +\infty$. Then

$$\frac{1}{T} \int_a^T \int_a^t \operatorname{tr} Z^2(s) ds dt \rightarrow \infty \quad \text{as } T \rightarrow +\infty.$$

From (5.8), it follows that $(1/T) \int_a^T \operatorname{tr} Z(s) ds \rightarrow \infty$ as $T \rightarrow \infty$, and so for large T we have, again using (5.8),

$$(5.9) \quad \frac{1}{T} \int_a^T \int_a^t \operatorname{tr} Z^2(s) ds dt \leq \frac{1}{T} \int_a^T \operatorname{tr} Z(s) ds + M \leq \frac{2}{T} \int_a^T \operatorname{tr} Z(s) ds.$$

Now by the Cauchy-Schwarz inequality we have

$$\left| \frac{1}{T} \int_a^T \operatorname{tr} Z(s) ds \right| \leq \left(\frac{1}{T} \int_a^T (\operatorname{tr} Z(s))^2 ds \right)^{1/2} \leq \left(\frac{n}{T} \int_a^T \operatorname{tr} Z^2(s) ds \right)^{1/2}$$

so that (5.9) gives

$$(5.10) \quad \left(\frac{1}{T} \int_a^T \int_a^t \operatorname{tr} Z^2(s) ds dt \right)^2 \leq \frac{4n}{T} \int_a^T \operatorname{tr} Z^2(s) ds.$$

If we set $H(T) \equiv \int_a^T \int_a^t \operatorname{tr} Z^2(s) ds dt$, we have from (5.10) that

$$(5.11) \quad H^2(t) \leq 4ntH'(t), \quad t \geq T_1,$$

and so

$$(5.12) \quad \frac{1}{4nt} \leq \frac{H'(t)}{H^2(t)}, \quad t \geq T_1.$$

Now an integration of (5.12) over $[T_1, \infty)$ gives

$$(5.13) \quad \infty = \frac{1}{4n} \ln t \Big|_{T_1}^{\infty} \leq \frac{1}{H(T_1)} < +\infty,$$

a contradiction. Thus $\lim_{t \rightarrow \infty} \int_a^t \operatorname{tr} Z^2(s) ds$ exists as a finite limit.

We see that this implies the existence of $\lim_{t \rightarrow \infty} \int_a^t Z^2(s) ds$, as follows: let the (operator) norm of a matrix A be denoted by $|A|$. For $a \leq s \leq t$, define $A(s, t)$ by $A(s, t) = \int_a^t Z^2(\sigma) d\sigma$. Then $A(s, t)$ is a nonnegative definite matrix and $|A(s, t)|^a = \lambda_1(A(s, t)) \leq \operatorname{tr} A(s, t) = \int_s^t \operatorname{tr} Z^2(\sigma) d\sigma$. This last integral converges to zero as $s, t \rightarrow \infty$ and so we have $|A(s, t)| \rightarrow 0$ as $s, t \rightarrow \infty$, i.e. $\int_s^t Z^2(\sigma) d\sigma \rightarrow 0$ as $s, t \rightarrow \infty$, yielding the existence of $\lim_{t \rightarrow \infty} \int_a^t Z^2(s) ds$ as asserted. This completes the proof of the lemma.

PROOF OF THEOREM 2.1. (A) Assume $L(Q) > -\infty$ and that

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \int_a^T \lambda_1 \left(\int_a^t Q(s) ds \right) dt = +\infty.$$

Suppose there exists a prepared solution $Y(t)$ of (1.1) which is not oscillatory. Without loss of generality, we may suppose that $\det Y(t) \neq 0$, $t \geq a$, so that from (2.2) we have (with $Z = -Y'Y^{-1}$)

$$(5.14) \quad \lambda_1(Z(t) - Z(a)) = \lambda_1 \left(\int_a^t Q(s) ds + \int_a^t Z^2(s) ds \right).$$

By the convexity of λ_1 and the fact that $\int_a^t Z^2(s) ds \geq 0$ (i.e., nonnegative definite for $t \geq a$), we have from (5.14) that $\lambda_1(Z(t) + \lambda_1(-Z(a))) \geq \lambda_1(\int_a^t Q(s) ds)$ and hence

$$(5.15) \quad \frac{1}{T} \int_a^T \lambda_1(Z(s)) ds + \frac{T-a}{T} \lambda_1(-Z(a)) \geq \frac{1}{T} \int_a^T \lambda_1 \left(\int_a^t Q(s) ds \right) dt$$

so that from hypothesis (A), there exists a sequence $T_n \rightarrow \infty$ with

$$(5.16) \quad \frac{1}{T_n} \int_a^{T_n} \lambda_1(Z(s)) ds \rightarrow +\infty \quad \text{as } T_n \rightarrow \infty.$$

Since $\lambda_1(Z(s))^2 \leq \lambda_1(Z^2(s))$ we have, by the Cauchy-Schwarz inequality

$$(5.17) \quad \begin{aligned} \left| \frac{1}{T_n} \int_a^{T_n} \lambda_1(Z(s)) ds \right| &\leq \left(\frac{1}{T_n} \int_a^{T_n} (\lambda_1(Z(s)))^2 ds \right)^{1/2} \\ &\leq \left(\frac{1}{T_n} \int_a^{T_n} \lambda_1(Z^2(s)) ds \right)^{1/2} \rightarrow +\infty \end{aligned}$$

as $T_n \rightarrow +\infty$. But from Lemma 5.1, since $L(Q) > -\infty$ it follows that $\int_a^\infty \lambda_1(Z^2(s)) ds \leq \int_a^\infty \operatorname{tr} Z^2(s) ds < +\infty$, so this contradiction proves part (A).

(B) Assume $L(Q) > -\infty$ and that

$$\limsup \frac{1}{T} \int_a^T \left[\lambda_1 \left(\int_a^t Q(s) ds \right) \right]^2 dt = +\infty.$$

As in part (A) above, we may assume $Z(t) = -Y'Y^{-1}$ is a solution of (2.2) for $t \geq a$ and so by Lemma 5.1, we obtain

$$(5.18) \quad Z(t) + \int_t^\infty Z^2(s) ds + C = \int_a^t Q(s) ds,$$

where $C = -Z(a) - \int_a^\infty Z^2(s) ds$. Hence, we have

$$(5.19) \quad \begin{aligned} \left(\lambda_1 \left(\int_a^t Q(s) ds \right) \right)^2 &= \left(\lambda_1 \left[Z(t) + \int_t^\infty Z^2(s) ds + C \right] \right)^2 \\ &\leq \lambda_1 \left(Z(t) + \int_t^\infty Z^2(s) ds + C \right)^2 \\ &\leq \text{tr} \left(Z(t) + \int_t^\infty Z^2(s) ds + C \right)^2 \\ &\leq 2 \text{tr} \left(Z(t) + \int_t^\infty Z^2(s) ds \right)^2 + 2 \text{tr} C^2 \leq 4 \text{tr} Z^2(t) \\ &\quad + 4 \text{tr} \left(\int_t^\infty Z^2(s) ds \right)^2 + 2 \text{tr} C^2, \end{aligned}$$

(since $\text{tr}(A+B)^2 \leq 2(\text{tr} A^2 + \text{tr} B^2)$). Therefore, we get

$$(5.20) \quad \begin{aligned} \frac{1}{T} \int_a^T \left[\lambda_1 \left(\int_a^t Q(s) ds \right) \right]^2 dt &\leq \frac{4}{T} \int_a^T \text{tr} \left(\int_t^\infty Z^2(s) ds \right)^2 dt \\ &\quad + \frac{4}{T} \int_a^T \text{tr} Z^2(t) dt + \frac{2(T-a)}{T} \text{tr} C^2. \end{aligned}$$

If we set $B(t) = \int_t^\infty Z^2(s) ds$, then $\text{tr} B(t) \rightarrow 0$ and $B(t) \geq 0$ so that $\lambda_1(B(t)) \rightarrow 0$ and hence $\lambda_1(B^2(t)) \rightarrow 0$. Therefore, $\text{tr} B^2(t) \rightarrow 0$ as $t \rightarrow \infty$. Thus, the first and second integral on the right side of (5.20) tend to 0 as $T \rightarrow +\infty$ and the last term is bounded. However, condition (B) implies that the left side of (5.20) is not bounded and this contradiction completes the proof of part (B).

(C) Assume $L(Q) > -\infty$ and $\limsup_{T \rightarrow \infty} \lambda_1(\int_a^T Q(s) ds) = +\infty$. As in part (B), we obtain equation (5.18) so that

$$(5.21) \quad \lambda_1(Z(t)) + \int_t^\infty \lambda_1(Z^2(s)) ds \geq \lambda_1 \left(\int_a^t Q(s) ds \right) + \lambda_n(-C).$$

Since $\int_t^\infty \lambda_1(Z^2(s)) ds \rightarrow 0$ as $t \rightarrow \infty$, it follows that $\int_t^\infty [\lambda_1(Z(s))]^2 ds \rightarrow 0$ as $t \rightarrow \infty$ and $\lambda_1(\int_t^\infty Z^2(s) ds) \rightarrow 0$ as $t \rightarrow \infty$. Now for any $k \geq 1$,

$$\mu \left\{ t: \lambda_1 \left(\int_a^t Q(s) ds \right) \geq k \right\} = +\infty$$

so that if $k \geq |\lambda_n(-C)| + 1$, then from (5.21)

$$(5.22) \quad \mu \left\{ t: \lambda_1(Z(t)) + \lambda_1 \left(\int_t^\infty Z^2(s) ds \right) \geq 1 \right\} = +\infty.$$

Since $\lambda_1(\int_t^\infty Z^2(s) ds) < \frac{1}{2}$ if $t \geq T_0$, say, we see that

$$(5.23) \quad \mu \{ t: \lambda_1(Z(t)) \geq \frac{1}{2} \} = +\infty.$$

That is, $\int_{E_k} (\lambda_1(Z(t)))^2 dt = +\infty$, where $E_k = \{t: \lambda_1(Z(t)) \geq \frac{1}{2}\}$. This contradiction proves part (C).

(D) Assume $L(Q) > -\infty$ and $\liminf_{T \rightarrow \infty} \lambda_1(\int_a^T Q(s) ds) = -\infty$. The proof is similar to part (C). Since for any $M > 0$,

$$(5.24) \quad \mu \left\{ t: \lambda_1 \left(\int_a^t Q(s) ds \right) < -M \right\} = +\infty$$

and since

$$\lambda_1(Z(t)) \leq \lambda_1 \left(Z(t) + \int_t^\infty Z^2(s) ds \right) \leq \lambda_1 \left(\int_a^t Q(s) ds \right) + \lambda_1(-C)$$

it follows that if $M > 1 + |\lambda_1(-C)|$, we have $\mu\{t: \lambda_1(Z(t)) \leq -1\} = +\infty$ so that $\int_a^\infty (\lambda_1(Z(t)))^2 dt = +\infty$, a contradiction. This completes the proof of Theorem 2.1.

PROOF OF THEOREM 2.2. Assume $L(Q) = -\infty$ and

$$\limsup_{t \rightarrow \infty} \lambda_n \left(\int_a^t Q(s) ds \right) = m > -\infty.$$

If the equation is nonoscillatory, then without loss of generality there is a solution Y of (1.1) for which $\det Y(t) \neq 0$ on $[a, \infty)$. Then with $Z = -Y'Y^{-1}$, $t \geq a$, we get

$$(5.26) \quad Z(t) = Z(a) + \int_a^t Z^2(s) ds + \int_a^t Q(s) ds.$$

By Lemma 5.1, since $L(Q) = -\infty$, it follows that $\text{tr} \int_a^t Z^2(s) ds \rightarrow +\infty$ as $t \rightarrow +\infty$ and hence $\lambda_1(\int_a^t Z^2(s) ds) \rightarrow +\infty$ as $t \rightarrow +\infty$. Since $\lambda_1(-\int_a^t Q(s) ds) = -\lambda_n(\int_a^t Q(s) ds)$ we have from (5.26)

$$(5.27) \quad \begin{aligned} \lambda_1 \left(Z(t) - \int_a^t Q(s) ds \right) &\leq \lambda_1(Z(t)) + \lambda_1 \left(-\int_a^t Q(s) ds \right) \\ &= \lambda_1(Z(t)) - \lambda_n \left(\int_a^t Q(s) ds \right). \end{aligned}$$

Now for any $\varepsilon > 0$, $\mu\{t: \lambda_n(\int_a^t Q(s) ds) \geq m - \varepsilon\} = +\infty$. From (5.26) we have

$$(5.28) \quad \begin{aligned} \frac{1}{n} \text{tr} \int_a^t Z^2(s) ds &= \frac{1}{n} \text{tr} \left(Z(t) - \int_a^t Q(s) ds - Z(a) \right) \\ &= \frac{1}{n} \text{tr} \left(Z(t) - \int_a^t Q(s) ds \right) - \frac{1}{n} \text{tr} Z(a) \\ &\leq \lambda_1 \left(Z(t) - \int_a^t Q(s) ds \right) - \frac{1}{n} \text{tr} Z(a) \\ &\leq \lambda_1(Z(t)) + \lambda_1 \left(-\int_a^t Q(s) ds \right) - \frac{1}{n} \text{tr} Z(a) \\ &= \lambda_1(Z(t)) - \lambda_n \left(\int_a^t Q(s) ds \right) - \frac{1}{n} \text{tr} Z(a) \end{aligned}$$

and since $(1/n) \operatorname{tr} \int_a^t Z^2(s) ds \geq (1/n) \int_a^t \lambda_1(Z^2(s)) ds$ we have from (5.28)

$$(5.29) \quad \frac{1}{n} \operatorname{tr} Z(a) + \frac{1}{n} \int_a^t \lambda_1(Z^2(s)) ds \leq \lambda_1(Z(t)) - \lambda_n \left(\int_a^t Q(s) ds \right), \quad t \geq a.$$

Hence, we have that for any $\varepsilon > 0$

$$(5.30) \quad \mu \left\{ t: \frac{1}{n} \operatorname{tr} Z(a) + \frac{1}{n} \int_a^t \lambda_1(Z^2(s)) ds \leq \lambda_1(Z(t)) - m + \varepsilon \right\} = +\infty$$

and since $\int_a^t \lambda_1(Z^2(s)) ds \geq \lambda_1(\int_a^t Z^2(s)) ds \rightarrow \infty$ as $t \rightarrow +\infty$, we see that if E is defined by

$$E = \left\{ t: \frac{1}{2n} \int_a^t \lambda_1(Z^2(s)) ds \leq \lambda_1(Z(t)) \right\} \cap [a+1, \infty)$$

then $\mu(E) = +\infty$. But now with $P(t) \equiv \int_a^t \lambda_1(Z^2(s)) ds$, we have $P'(t) = \lambda_1(Z^2(t)) \geq \lambda_1(Z(t))^2$ and so $P'(t) \geq P^2(t)/4n^2$, $t \in E$, and now $\int_E P'(t)/P^2(t) dt \geq \mu(E)/4n^2 = +\infty$, a contradiction since the integral on the left is $\leq 1/P(a+1)$. This completes the proof of Theorem 2.2.

Before proving Theorem 2.3, we require a lemma.

LEMMA 5.2. *Let $p(t)$ be locally bounded, nonnegative and measurable on $[a, \infty)$ with $p(t)$ not almost everywhere zero. Let $q(t)$ be nonnegative and locally integrable such that*

$$(5.31) \quad \int_a^t p(s) ds \geq q(t) \int_a^t \int_a^s p^2(\sigma) d\sigma ds, \quad \text{for almost all } t \geq a.$$

Then for all sufficiently large $\bar{a} > a$, $t^{-1/2}q(t) \in L^2[\bar{a}, \infty)$.

PROOF. Let $P(t) = \int_a^t \int_a^s p^2(\sigma) d\sigma ds$. Then P is continuously differentiable and $P(t) > 0$ for $t > a^*$, say. We have $P'(t) = \int_a^t p^2(s) ds$ and by the Cauchy-Schwarz inequality

$$(5.32) \quad P'(t) \geq (t-a)^{-1} \left(\int_a^t p(s) ds \right)^2 \geq (t-a)^{-1} q^2(t) P^2(t).$$

Since $\int_{\bar{a}}^\infty P'(t)/P^2(t) dt < \infty$ for $\bar{a} > a^*$, the result follows.

PROOF OF THEOREM 2.3. The argument is similar to that given in the proof of Theorem 3.1 of [4], so we content ourselves here with giving a sketch of the proof. If (1.1) is nonoscillatory, we may, without loss of generality, find a prepared solution Y with $\det Y(t) \neq 0$ on $[a, \infty)$. If $Z = -Y'Y^{-1}$, then Z is symmetric and we have

$$(5.33) \quad Z(t) = \int_a^t Z^2(s) ds = \int_a^t Q(s) ds + C, \quad t \geq a,$$

for some constant matrix C .

It is known that for any continuous $n \times n$ symmetric matrix-valued function, a continuously varying orthonormal system may be selected [22]. Consequently, we may choose a locally integrable vector function $x(t)$ with $\|x(t)\| = 1$ such that

$$(5.34) \quad x^*(t) \left(\int_a^t Q(s) ds \right) x(t) = \lambda_1 \left(\int_a^t Q(s) ds \right)$$

(possible discontinuities in $x(t)$ arise because of the coalescence of eigenvalues).

Let the eigenvalues of $Z(t)$ be $\mu_1(t) \geq \dots \geq \mu_n(t)$. By the preceding remark, we may select a system of orthonormal locally integrable eigenvectors $e_i(t)$ such that

$$(5.35) \quad Z(t)e_i(t) = \mu_i(t)e_i(t); \quad e_i^*(t)e_j(t) = \delta_{ij}.$$

Let $U(t)$ be the orthogonal matrix whose rows are the $e_i(t)$, and let $c_i(s, t)$ be the projection of $x(t)$ onto $e_i(s)$. Let the components of $U^{-1}(t)c(a, t)$ be $v_i(t)$, and let the components of $(U^{-1}(s) - U^{-1}(t))c(a, t)$ be $w_i(s, t)$, $i = 1, \dots, n$. Then if we denote the left-hand side of (5.33) by $\Phi(t)$, we have (see [4])

$$(5.36) \quad \lambda_1 \left(\int_a^t Q(s) ds \right) + x^*(t)Cx(t) = x^*(t)\Phi(t)x(t) \\ = \sum_{i=1}^n \mu_i(t)v_i^2(t) - \int_a^t \sum_{i=1}^n \mu_i^2(s)(v_i(t) + w_i(s, t))^2 ds.$$

The arguments given in [4] show that hypothesis (2.7) allows us to find a unit vector $\hat{x}(t)$, functions $\hat{\mu}_{ij}(t)$; $i = 1, \dots, n$, $j = 1, \dots, m$, and $\delta > 0$ such that

$$(5.37) \quad \frac{1}{3} \lambda_1 \left(\int_a^t Q(s) ds \right) - 2\|C\| \leq \hat{x}^*(t)\Phi(t)\hat{x}(t) \\ \leq \sum_{ij} \left(\hat{\mu}_{ij}(t) - \delta \int_a^t \hat{\mu}_{ij}(s) ds \right).$$

Hypothesis (A) and (5.37) imply that there exists $t_1 > a$ such that

$$(5.38) \quad \int_a^t \sum_{i,j} \left(\hat{\mu}_{ij}(s) - \delta \int_a^s \hat{\mu}_{ij}^2(\sigma) d\sigma \right) ds > 0, \quad t \geq t_1.$$

If $\hat{\mu}_{ij} \equiv 0$ a.e. for $t \geq a$, let $\hat{q}_{ij}(t) = 0$; otherwise define $\hat{q}_{ij}(t)$ by

$$(5.39) \quad \int_a^t |\hat{\mu}_{ij}(s)| ds = \hat{q}_{ij}(t) \int_a^t \int_a^s |\mu_{ij}|^2(\sigma) d\sigma ds.$$

By Lemma 5.2, each \hat{q}_{ij} has the property that $t^{-1/2}\hat{q}_{ij}(t) \in L^2[\bar{a}, \infty)$ if $\bar{a} > t_1$ is sufficiently large, and so $\hat{q}(t) = \sum_{i,j} \hat{q}_{ij}(t) \in L^2[\bar{a}, \infty)$, if \bar{a} is sufficiently large. However, (5.38) implies that

$$(\hat{q}(t) - \delta) \sum_{i,j} \int_a^t \int_a^s \hat{\mu}_{ij}(\sigma) d\sigma ds > 0, \quad t > \bar{a},$$

and so $\hat{q}(t) > \delta$ for $t > \bar{a}$, contradicting the square-integrability of $t^{-1/2}\hat{q}(t)$. This proves the theorem.

6. Proofs for §3.

PROOF OF THEOREM 3.1. Choose any sequence $a_k > a$ with $\lim_{k \rightarrow \infty} a_k = \infty$. We have

$$\lambda_1 \left(\int_{a_k}^t Q(s) ds \right) \geq \lambda_1 \left(\int_a^t Q(s) ds \right) - \lambda_1 \left(\int_a^{a_k} Q(s) ds \right)$$

and so, by (3.2), $\limsup_{t \rightarrow \infty} \lambda_1(\int_{a_k}^t Q(s) ds) = \infty$. Therefore we may choose $b_k > a_k + 2$ and $x_k \in \mathbf{R}^n$ with $\|x_k\| = 1$, so that

$$(6.1) \quad x_k^* \left(\int_{a_k}^{b_k} Q(s) ds \right) x_k \geq k, \quad k = 1, 2, \dots$$

Suppose that condition (A) holds. Then there exists M such that $\lambda_1(Q(t)) \leq M$, $t \geq a$. Define $\varphi_k \in A_1(a_k, b_k)$ as follows:

$$(6.2) \quad \varphi_k(t) = \begin{cases} (t - a_k)x_k, & a_k \leq t \leq a_k + 1, \\ x_k, & a_k + 1 \leq t \leq b_k - 1, \\ (-t + b_k)x_k, & b_k - 1 \leq t \leq b_k. \end{cases}$$

We have $\int_{a_k}^{b_k} |\varphi'_k(t)|^2 dt = 2$, and

$$(6.3) \quad \begin{aligned} \int_{a_k}^{b_k} \varphi_k^*(t) Q(t) \varphi_k(t) dt &= \int_{a_k}^{b_k} x_k^* Q(t) x_k dt - \int_{a_k}^{a_k+1} [1 - (t - a_k)^2] x_k^* Q(t) x_k dt \\ &\quad - \int_{b_k-1}^{b_k} [1 - (b_k - t)^2] x_k^* Q(t) x_k dt \geq k - \frac{4}{3}M. \end{aligned}$$

Thus

$$\int_{a_k}^{b_k} \{|\varphi'_k(t)|^2 - \varphi_k^*(t) Q(t) \varphi_k(t)\} dt \leq 2 + \frac{4}{3}M - k < 0$$

for k sufficiently large. It follows that (1.1) is oscillatory.

If condition (B) holds, we have $\lambda_n(Q(t)) \geq -M$, $t \geq a$, and we define $\varphi_k \in A_1(a_k - 1, b_k + 1)$ by

$$(6.4) \quad \varphi_k(t) = \begin{cases} (t + 1 - a_k)x_k, & a_k - 1 \leq t \leq a_k, \\ x_k, & a_k \leq t \leq b_k, \\ (-t + b_k + 1)x_k, & b_k \leq t \leq b_k + 1. \end{cases}$$

Again $\int_{a_k-1}^{b_k+1} |\varphi'_k(t)|^2 dt = 2$, and we have

$$(6.5) \quad \begin{aligned} \int_{a_k-1}^{b_k+1} \varphi_k^*(t) Q(t) \varphi_k(t) dt &= \int_{a_k-1}^{a_k} \varphi_k^*(t) Q(t) \varphi_k(t) dt \\ &\quad + \int_{b_k}^{b_k+1} \varphi_k^*(t) Q(t) \varphi_k(t) dt + \int_{a_k}^{b_k} \varphi_k^*(t) Q(t) \varphi_k(t) dt \\ &= \int_{a_k}^{b_k} x_k^* Q(t) x_k dt + \int_{a_k-1}^{a_k} (t + 1 - a_k)^2 x_k^* Q(t) x_k dt \\ &\quad + \int_{b_k}^{b_k+1} (b_k + 1 - t)^2 x_k^* Q(t) x_k dt \\ &\geq k - M \left\{ \int_{a_k-1}^{a_k} (t + 1 - a_k)^2 dt + \int_{b_k}^{b_k+1} (b_k + 1 - t)^2 dt \right\} \\ &= k - \frac{2}{3}M \end{aligned}$$

and so

$$\int_{b_k}^{b_k+1} \{|\varphi'_k(t)|^2 - \varphi_k^*(t)Q(t)\varphi_k(t)\} dt \leq 2 + \frac{2}{3}M - k < 0$$

for k sufficiently large, and again we conclude that (1.1) is oscillatory.

PROOF OF THEOREM 3.2. Applying the techniques of the previous theorem to the scalar equation

$$(6.6) \quad y''(t) + q_{ij}(t)y(t) = 0$$

there exist intervals $[a_m, b_m]$ with $\lim_{m \rightarrow \infty} a_m = \infty$ and a scalar function $\varphi_m \in A_1(a_m, b_m)$ (with $n = 1$) such that

$$(6.7) \quad \int_{a_m}^{b_m} \{|\varphi'_m(t)|^2 - q_{ij}(t)|\varphi_m(t)|^2\} dt < 0.$$

Define $\psi_m \in A_1(a_m, b_m)$ by

$$\psi_m(t) = \text{col}(0, \dots, \varphi_m(t), 0, \dots, \varphi_m(t), \dots, 0)$$

where the only nonzero entries are the i th and j th. Then

$$\int_{a_m}^{b_m} \{|\psi'_m(t)|^2 - \psi_m^*(t)Q(t)\psi_m(t)\} dt = 2 \int_{a_m}^{b_m} \{|\varphi'_m(t)|^2 - q_{ij}(t)|\varphi_m(t)|^2\} dt < 0$$

implying oscillation for (1.1). A similar argument was used in [9].

PROOF OF THEOREM 3.3. Let $m \geq a + 1$ be a positive integer. Without loss of generality, we may assume that $0 < \varepsilon_m < 1$. Let

$$(6.8) \quad \max_{m-\varepsilon_m/2 \leq t \leq m} \|Q(t)\| = \alpha_m$$

and choose the positive integer $k = k_m$ so that

$$(6.9) \quad k > 2 + \frac{1}{12}\alpha_m\varepsilon_m^2.$$

Let x_{mk} be the unit vector given by the hypothesis of the theorem. Since $\mu(S_{mk}) \geq \varepsilon_m$, we may find a finite collection of closed intervals $I_j = [a_j, b_j]$, $1 \leq j \leq r$, where $m < a_1 < b_1 < \dots < a_r < b_r$, such that

$$\bigcup_{j=1}^r I_j \subset S_{mk} \quad \text{and} \quad \mu\left(\bigcup_{j=1}^r I_j\right) = \sum_{j=1}^r \delta_j \geq \frac{1}{2}\varepsilon_m,$$

where $\delta_j = b_j - a_j$. Since $\int_m^t Q(s) ds$ is a continuous function of t , we may, by maximizing the lengths of the intervals, assume that

$$(6.10) \quad x_{mk}^* \left(\int_m^{a_j} Q(s) ds \right) x_{mk} = k, \quad 1 \leq j \leq r,$$

$$(6.11) \quad x_{mk}^* \left(\int_m^{b_j} Q(s) ds \right) x_{mk} = k, \quad 1 \leq j \leq r-1.$$

Furthermore, by discarding intervals if necessary, and shrinking the final interval, we may suppose that

$$(6.12) \quad \sum_{j=1}^r \delta_j = \frac{1}{2}\varepsilon_m.$$

Let a_0 and b_0 be defined to be m , δ_0 to be 0, and let $\Delta_j = \sum_{i=0}^j \delta_i$, $j = 0, 1, \dots, r$.

We define $\varphi_m \in A_1(m - \frac{1}{2}\varepsilon_m, b_r)$ as follows:

$$(6.13) \quad \varphi_m(t) = \rho_m(t)x_{mk},$$

where

$$(6.14) \quad \rho_m = \begin{cases} t + \frac{1}{2}\varepsilon_m - m, & m - \frac{1}{2}\varepsilon_m \leq t \leq m, \\ \frac{1}{2}\varepsilon_m - \Delta_j, & b_j \leq t \leq a_{j+1}; \quad 0 \leq j \leq r-1, \\ \frac{1}{2}\varepsilon_m - \Delta_{j-1} + a_j - t, & a_j \leq t \leq b_{j+1}; \quad 0 \leq j \leq r. \end{cases}$$

We have

$$(6.15) \quad \int_{m-\varepsilon_m/2}^{b_r} |\varphi'_m(t)|^2 dt = \frac{1}{2}\varepsilon_m + \Delta_r = \varepsilon_m$$

and

$$(6.16) \quad \begin{aligned} & \int_{m-\varepsilon_m/2}^{b_r} \varphi_m^*(t)Q(t)\varphi_m(t) dt \\ &= \left(\int_{m-\varepsilon_m/2}^{b_r} + \sum_{j=0}^{r-1} \int_{b_j}^{a_{j+1}} + \sum_{j=1}^{r-1} \int_{a_j}^{b_j} \right) \varphi_m^*(t)Q(t)\varphi_m(t) dt. \end{aligned}$$

By (6.10) and (6.11),

$$(6.17) \quad \begin{aligned} \int_{m-\varepsilon_m/2}^m \varphi_m^*(t)Q(t)\varphi_m(t) dt &\geq -\alpha_m \int_{m-\varepsilon_m/2}^m \left(t + \frac{1}{2}\varepsilon_m - m \right)^2 dt \\ &= -\frac{1}{24}\alpha_m \varepsilon_m^3. \end{aligned}$$

$$(6.18) \quad \begin{aligned} \int_{b_0}^{a_1} \varphi_m^*(t)Q(t)\varphi_m(t) dt &= \frac{1}{2}\varepsilon_m \int_m^{a_1} x_{mk}^*Q(t)x_{mk} dt \\ &\geq \frac{1}{2}\varepsilon_m x_{mk}^* \left(\int_m^{a_1} Q(t) dt \right) x_{mk} = \frac{1}{2}\varepsilon_m k, \end{aligned}$$

by (6.10).

For $1 \leq j \leq r-1$,

$$\begin{aligned} \int_{b_j}^{a_{j+1}} \varphi_m^*(t)Q(t)\varphi_m(t) dt &= \left(\frac{1}{2}\varepsilon_m - \Delta_j \right) \int_{b_j}^{a_{j+1}} x_{mk}^*Q(t)x_{mk} dt \\ &= \left(\frac{1}{2}\varepsilon_m - \Delta_j \right) \int_m^{a_j} x_{mk}^*Q(t)x_{mk} dt - \int_m^{b_j} x_{mk}^*Q(t)x_{mk} dt = 0 \end{aligned}$$

by (6.10) and (6.11).

Noting that for $1 \leq j \leq r$, $\rho_m(t) \geq 0$, and $\rho'_m(t) \equiv -1$ on $[a_j, b_j]$, we have, on integrating by parts,

$$(6.20) \quad \begin{aligned} \int_{a_j}^{b_j} \varphi_m^*(t)Q(t)\varphi_m(t) dt &= \int_{a_j}^{b_j} \rho_m^2(t)x_{mk}^*Q(t)x_{mk} dt \\ &= \rho_m^2(b_j) \int_{a_j}^{b_j} x_{mk}^*Q(t)x_{mk} dt + \int_{a_j}^{b_j} 2\rho_m(t) \left(\int_{a_j}^t x_{mk}^*Q(s)x_{mk} ds \right) dt. \end{aligned}$$

By (6.10) and (6.11),

$$(6.21) \quad \int_{a_j}^{b_j} x_{mk}^* Q(t) x_{mk} dt = \left(\int_m^{b_j} - \int_m^{a_j} \right) x_{mk}^* Q(t) x_{mk} dt \geq 0 \quad (\text{with equality if } 1 \leq j \leq r-1).$$

Since $t \in [a_j, b_j] \Rightarrow t \in S_{mk}$, using (6.10) we find that for $t \in [a_j, b_j]$,

$$(6.22) \quad \int_{a_j}^t x_{mk}^* Q(s) x_{mk} ds = \left(\int_m^t - \int_m^{a_j} \right) x_{mk}^* Q(s) x_{mk} ds \geq 0.$$

From (6.20)–(6.22), we have

$$(6.23) \quad \int_{a_j}^{b_j} \varphi_m^*(t) Q(t) \varphi_m(t) dt \geq 0, \quad j = 1, \dots, r.$$

From (6.15)–(6.19) and (6.23), we see that

$$(6.24) \quad \int_{m-\varepsilon_m/2}^{b_r} \{|\varphi'_m(t)|^2 - \varphi_m^*(t) Q(t) \varphi_m(t)\} dt \leq \varepsilon_m + \frac{1}{24} \alpha_m \varepsilon_m^3 - \frac{1}{2} \varepsilon_m k < 0,$$

by (6.9). Since $m - \frac{1}{2} \varepsilon_m > m - \frac{1}{2}$ and m is an arbitrary positive integer exceeding a , the theorem follows.

PROOF OF THEOREM 3.4. Since (3.6) is oscillatory, there exist subintervals $[\tilde{a}_m, \tilde{b}_m]$ of $[a, A]$ with $\lim_{m \rightarrow \infty} \tilde{a}_m = A$, and $\tilde{\varphi}_m \in (\tilde{a}_m, \tilde{b}_m)$ such that

$$(6.25) \quad \int_{\tilde{a}_m}^{\tilde{b}_m} \{|\tilde{\varphi}'_m(t)|^2 - \tilde{\varphi}_m^*(t) \tilde{Q}(t) \tilde{\varphi}_m(t)\} dt < 0, \quad m = 1, 2, \dots$$

Choose $a_m, b_m \geq a$ so that if Q_m is the restriction of Q to $[a_m, b_m)$ and \tilde{Q}_m is the restriction of \tilde{Q} to $[\tilde{a}_m, \tilde{b}_m)$, and if $S_m = S \cap [a_m, b_m)$, then $\tilde{Q}_m = T_{S_m}(Q_m)$. Then $\lim_{m \rightarrow \infty} a_m = \infty$. Let φ_m be defined on $[a_m, b_m)$ so that φ_m is continuous and $\tilde{\varphi}_m = T_{S_m}(\varphi_m)$. Extend φ_m to $[a_m, b_m]$ by defining $\varphi_m(b_m) = 0$. Then $\varphi_m \in A_1(a_m, b_m)$ and φ_m is constant on each component interval of $[a_m, b_m) \setminus S_m$. We have

$$(6.26) \quad \int_{a_m}^{b_m} \{|\varphi'_m(t)|^2 - \varphi_m^*(t) Q(t) \varphi_m(t)\} dt = \int_{\tilde{a}_m}^{\tilde{b}_m} \{|\tilde{\varphi}'_m(t)|^2 - \tilde{\varphi}_m^*(t) \tilde{Q}(t) \tilde{\varphi}_m(t)\} dt - \sum \int_{\alpha_i}^{\beta_i} \varphi_m^*(t) Q(t) \varphi_m(t) dt$$

where the summation is over indices i for which $(\alpha_i, \beta_i) \subset [a_m, b_m)$. By (6.25) and the nonnegative definiteness condition (3.5), the right-hand side of (6.26) is negative. Thus (1.1) is oscillatory.

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